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Torsional oscillations of a non-Newtonian fluid with a free surface

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A study is made of the flow and the shape of the free surface of a non-Newtonian fluid contained in a flat-bottomed cylindrical vessel performing torsional oscillations of small amplitude. For the case where the fluid depth is small compared with the vessel radius, the solution is shown to have a simple radial dependence except in a boundary-layer region near the side wall. It is shown that under certain circumstances the mean steady second-order components of both free surface curvature and radial–axial flow may be in the reverse direction to those for a Newtonian fluid. It is found that flow reversal may occur at any frequency of oscillation, but that surface curvature reversal cannot occur at low frequencies.

1. Introduction

Suppose that a flat-bottomed cylindrical vessel, partially filled with non-Newtonian fluid, is made to perform torsional oscillations about its axis, which is vertical. In this paper we examine the nature of the flow induced in the fluid by the motion of the container; in particular we compute the resulting deformation of the free surface.

A related problem, the calculation of the flow field between two infinite parallel planes, one of which is performing torsional oscillations, was studied by Rosenblat (1978). In that paper it was shown that, under certain circumstances, the direction of the mean second-order flow was opposite to that which would occur in a Newtonian fluid. We find a similar effect in the present configuration. In addition to this flow reversal we show that the fluid in the centre of the container may rise, instead of falling as it does in the Newtonian case. The rise of the fluid is a kind of Weissenberg effect, brought about by the action of normal stresses, except that the rise does not occur adjacent to a rotating rod.

We shall follow Rosenblat's treatment in several respects, and use the same constitutive equations. Modifications are needed, however, to take into account the free surface and the finite vessel radius. To make the problem more tractable we neglect surface tension and assume that the radius of the vessel is substantially greater than the fluid depth. This assumption leads to a two-parameter problem. One parameter represents the (small) amplitude of the torsional oscillations, while the second parameter is the ratio of depth to radius. Given that the latter is small we find that the flow in the vessel is described by a similarity solution except in an annular region near the side wall; the width of this region is of the order of the depth of the fluid.

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We calculate in detail the mean (steady) second-order flow in the central, core region. We show that under a certain set of conditions the direction of this flow is opposite to that in a Newtonian fluid, and that under a *different* set of conditions the free surface may rise at the centre instead of falling. Although both effects are due to the nonlinearity of the fluid, they appear at different levels of nonlinearity. We indicate the required conditions by evaluating asymptotic solutions for low and high frequencies, and by some numerical computations.

2. Formulation

The constitutive equations which we shall use were originally given by Coleman & Noll (1961) and were derived in the form we require by Joseph (1976). They describe a general simple fluid undergoing (not necessarily steady) motions of small amplitude. We let $\epsilon > 0$ be the parameter which measures the smallness of these motions, uniformly in space and time; the question of how small ϵ should be is postponed until later.

It is known (cf. Rosenblat 1978) that the stress tensor in the case of small ϵ can be written as

$$\mathbf{S} = \epsilon \mathbf{S}_1 + \epsilon^2 \mathbf{S}_2 + O(\epsilon^3) \tag{2.1}$$

where

$$\mathbf{S}_1 = \int_0^\infty \zeta(s)\, \mathbf{A}_1(s)\, ds \tag{2.2}$$

and

$$\mathbf{S}_2 = \int_0^\infty \zeta(s)\, \mathbf{A}_2(s)\, ds + \int_0^\infty \zeta(s) \{(\boldsymbol{\xi}_1 . \nabla)\, \mathbf{A}_1(s) + \mathbf{A}_1(s) . \nabla\boldsymbol{\xi}_1 + \nabla\boldsymbol{\xi}_1^T . \mathbf{A}_1(s)\}\, ds + \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, \mathbf{A}_1(s_1) . \mathbf{A}_1(s_2)\, ds_1\, ds_2. \tag{2.3}$$

Here we have expanded the velocity field in the form

$$\mathbf{u} = \epsilon \mathbf{u}_1 + \epsilon^2 \mathbf{u}_2 + O(\epsilon^3), \tag{2.4}$$

and we have defined

$$\mathbf{A}_i(s) = \nabla \mathbf{u}_i(t-s) + [\nabla \mathbf{u}_i(t-s)]^T, \quad i = 1, 2 \tag{2.5}$$

and

$$\boldsymbol{\xi}_1(s) = -\int_0^s \mathbf{u}_1(t-s')\, ds'. \tag{2.6}$$

All quantities are also dependent on a spatial variable $\mathbf{x}$; $\zeta(s)$ and $\gamma(s_1, s_2)$ are material functions, with $\gamma(s_1, s_2)$ symmetric in its arguments.

We shall construct the solution as a power series in ϵ up to second order, which, it turns out, is the lowest order at which the free surface is perturbed from its rest state. We assume that the motion is axisymmetric and use cylindrical polar co-ordinates (r, θ, z). The base of the vessel is located in the plane $z = 0$, and we assume the vessel to be a circular cylinder with side wall at $r = a$. The velocity field is then

$$\mathbf{u}(r, \theta, z, t) = u(r, z, t)\, \hat{\mathbf{r}} + v(r, z, t)\, \hat{\boldsymbol{\theta}} + w(r, z, t)\, \hat{\mathbf{z}}, \tag{2.7}$$

where $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{z}}$ are unit vectors in the co-ordinate directions. The free surface is given by $z = h(r, t)$ for $0 \leqslant r < a$. The equations of motion are, in standard notation,

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} . \nabla \mathbf{u} \right) = -\nabla \Phi + \nabla . \mathbf{S} \tag{2.8}$$

where

$$\Phi = p + \rho g z; \tag{2.9}$$

the continuity equation is
$$\nabla . \mathbf{u} = 0. \tag{2.10}$$
At the free surface we require a kinematic condition and conditions on the normal and tangential stress components. The former is easily shown to be
$$w = \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r}, \quad z = h(r,t), \tag{2.11}$$
while the stress conditions in the absence of surface tension are (cf. Joseph & Beavers 1976)
$$p_a - \Phi + S_{zz} - \frac{\partial h}{\partial r} S_{rz} + \rho g h = 0 \tag{2.12}$$
on $z = h(r, t)$ for the normal stress, and
$$S_{\theta z} = \left\{1 - \left(\frac{\partial h}{\partial r}\right)^2\right\} S_{rz} + (S_{zz} - S_{rr})\frac{\partial h}{\partial r} = 0 \tag{2.13}$$
on $z = h(r,t)$ for the tangential stress components. In (2.12) p_a is the atmospheric pressure.

We assume that the vessel is rotating with an angular velocity given by
$$\epsilon\omega \cos \omega t = \epsilon\omega \operatorname{Re} e^{i\omega t}. \tag{2.14}$$
(According to this definition the amplitude of the oscillation is ϵ radians.) It follows that
$$\mathbf{u} = \epsilon\omega r \operatorname{Re} e^{i\omega t}\hat{\boldsymbol{\theta}} \quad \text{when} \quad z = 0, \quad 0 \leqslant r < a, \tag{2.15}$$
and
$$\mathbf{u} = \epsilon\omega a \operatorname{Re} e^{i\omega t}\hat{\boldsymbol{\theta}} \quad \text{when} \quad r = a, \quad 0 \leqslant z \leqslant h(a,t), \tag{2.16}$$
are the boundary conditions at the bottom and the side of the vessel respectively. Finally we require a condition that the amount of fluid in the container is conserved:
$$\int_0^a h(r,t).\, 2\pi r\, dr = \pi a^2 d, \tag{2.17}$$
where d is a given constant.

In accordance with our proposed perturbation scheme we shall look for solutions for the quantities $\mathbf{S}(r,z,t;\, \epsilon)$, $\mathbf{u}(r,z,t;\, \epsilon)$, $\Phi(r,z,t;\, \epsilon)$ and $h(r,t;\, \epsilon)$ in the forms
$$\begin{aligned} \mathbf{u} &= \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + O(\epsilon^3), \quad \mathbf{S} = \epsilon\mathbf{S}_1 + \epsilon^2\mathbf{S}_2 + O(\epsilon^3), \\ \Phi &= \Phi_0 + \epsilon\Phi_1 + \epsilon^2\Phi_2 + O(\epsilon^3), \quad h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + O(\epsilon^3). \end{aligned} \tag{2.18}$$

At order zero there is no motion and in the absence of surface tension the surface is flat. We take the solution to be
$$h_0 = d, \quad \Phi_0 = p_a + \rho g d; \tag{2.19}$$
the first and second-order corrections to this equilibrium state are calculated in the next two sections.

3. First-order solution

Consideration of the effect of reversing the sign of ϵ shows that u, w and h will be even functions of ϵ, while v will be an odd function of ϵ. We expect therefore that the first-order solution will take the form
$$\mathbf{u}(r,z,t) = v_1(r,z,t)\,\hat{\boldsymbol{\theta}}, \quad \Phi_1 = 0, \quad h_1 = 0. \tag{3.1}$$
The boundary conditions (2.15) and (2.16) suggest the structure
$$v_1(r,z,t) = \operatorname{Re}\{V(r,z)\, e^{i\omega t}\}. \tag{3.2}$$

Then using (2.2) we find that

$$\nabla . \mathbf{S}_1 = \int_0^\infty \zeta(s)\, \nabla^2 \mathbf{u}_1(t-s)\, ds, \tag{3.3}$$

and so (2.8) gives at first order in ϵ

$$i\omega\rho V = \mu\left\{\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-\frac{V}{r^2}+\frac{\partial^2 V}{\partial z^2}\right\}, \tag{3.4}$$

where

$$\mu \equiv \int_0^\infty \zeta(s)\, e^{-i\omega s} ds. \tag{3.5}$$

We transform (3.4) into dimensionless form by putting $V = r\omega U$, $r = r'a$, $z = z'd$; (3.4) then becomes

$$\lambda^2\left(\frac{\partial^2 U}{\partial r^2}+\frac{3}{r}\frac{\partial U}{\partial r}\right)+\frac{\partial^2 U}{\partial z^2} = \Omega^2 U, \tag{3.6}$$

where the primes have been omitted, and where we have defined

$$\lambda = d/a, \quad \Omega = (i\omega\rho d^2/\mu)^{\frac{1}{2}}. \tag{3.7}$$

The solution of (3.6) must satisfy the boundary conditions (2.11)–(2.13) and (2.15)–(2.17) to the appropriate order in ϵ. The conditions (2.11)–(2.13) should be applied on the unknown free surface, but they can be applied on the plane $z = 1$ by using Taylor's theorem and the perturbation scheme (2.18). Since $h_1 = 0$ we see that (2.11) and (2.17) are satisfied identically, and since $u_1 = w_1 = 0$ the remaining conditions can easily be shown to reduce to

$$\partial U/\partial z = 0 \quad \text{on} \quad z = 1, \quad 0 \leqslant r \leqslant 1, \tag{3.8}$$

$$U = 1 \quad \text{on} \quad z = 0, \quad 0 \leqslant r \leqslant 1, \tag{3.9}$$

and

$$U = 1 \quad \text{on} \quad r = 1, \quad 0 \leqslant z \leqslant 1. \tag{3.10}$$

For sufficiently small λ equation (3.6) has the form of a singular perturbation problem which can be solved by the method of matched asymptotic expansions. We anticipate a boundary layer near the side wall $r = 1$ and an outer region away from the wall. The governing equation in the outer region is obtained by setting $\lambda = 0$ in (3.6):

$$\partial^2 U/\partial z^2 = \Omega^2 U, \tag{3.11}$$

and the solution of this is required to satisfy the conditions (3.8) and (3.9). We obtain

$$U = U_0(z) = \frac{\cosh \Omega(1-z)}{\cosh \Omega}. \tag{3.12}$$

To determine the solution near the wall we introduce the boundary-layer co-ordinate

$$\xi = (1-r)/\lambda \tag{3.13}$$

and set

$$U(r,z) = U_0(z) + U_1(\xi, z). \tag{3.14}$$

We find that to a first approximation

$$\frac{\partial^2 U_1}{\partial \xi^2}+\frac{\partial^2 U_1}{\partial z^2} = \Omega^2 U_1, \tag{3.15}$$

with boundary conditions

$$\frac{\partial U_1}{\partial z}(\xi, 1) = 0, \quad U_1(\xi, 0) = 0 \quad \text{and} \quad U_1(0,z) = 1 - U_0(z) \tag{3.16}$$

together with the requirement that $U_1(\xi, z) \to 0$ as $\xi \to \infty$. We obtain the solution

$$U_1(\xi, z) = \sum_{m=0}^{\infty} \frac{2\Omega^2}{(m+\frac{1}{2})\pi\Omega_m^2} \exp(-\Omega_m \xi) \sin(m+\tfrac{1}{2})\pi z, \tag{3.17}$$

where
$$\Omega_m = [\Omega^2 + (m+\tfrac{1}{2})^2\pi^2]^{\frac{1}{2}}. \tag{3.18}$$

Equation (3.14) now gives the composite flow solution for the first-order field. Away from the side wall, however, a good approximation is given by (3.12), which corresponds to a similarity solution with velocity proportional to radius.

4. Second-order steady flow

The nonlinear self-interaction of the first-order flow generates a second-order motion. It is clear from the structure of the problem that this second-order flow field will comprise a steady component and a second harmonic component. Hence we can write (in dimensional variables)

$$\mathbf{u}_2(r, z, t) = \mathbf{u}_2^{(0)}(r, z) + \mathrm{Re}\,[e^{2i\omega t}\mathbf{u}_2^{(2)}(r, z)], \tag{4.1}$$

and similarly for $\Phi_2(r, z, t)$, $\mathbf{S}_2(r, z, t)$ and $h_2(r, t)$. We shall confine our attention henceforth to the steady component of the flow, the 'steady streaming', and omit all considerations of the second harmonic.

We note first of all that the steady component of the second-order stress tensor $\mathbf{S}_2$, given by equation (2.3), can be written in the form

$$\mathbf{S}_2^{(0)} = \int_0^\infty \zeta(s)\,\mathbf{A}_2^{(0)}(s)\,ds + \mathbf{Z}^{(0)}, \tag{4.2}$$

where $\mathbf{Z}^{(0)}$ is the steady part of the second and third terms in (2.3). Thus $\mathbf{Z}^{(0)}$ represents the forcing due to the self-interaction of the first-order flow. From (2.5) we have that

$$\mathbf{A}_2^{(0)}(s) = \nabla\mathbf{u}_2^{(0)}(t-s) + [\nabla\mathbf{u}_2^{(0)}(t-s)]^T \tag{4.3}$$

and it follows that the equations of motion for the steady second-order flow can be written as

$$\mu_0\nabla^2\mathbf{u}_2^{(0)} - \nabla\Phi_2^{(0)} = \rho[\mathbf{u}_1.\nabla\mathbf{u}_1]^{(0)} - \nabla.\mathbf{Z}^{(0)}, \tag{4.4}$$

where
$$\mu_0 \equiv \int_0^\infty \zeta(s)\,ds. \tag{4.5}$$

The equations (4.4) and the continuity equation have to be solved subject to the appropriate boundary conditions.

Our procedure will be as follows. We shall solve the problem in the central (outer) region without imposing the side-wall boundary conditions, and then we shall consider what adjustments need to be made to the solution by way of a boundary-layer correction. To this end we seek a solution of the form

$$\mathbf{u}_2^{(0)} = \mathbf{u}_0 + \mathbf{u}_1, \quad \Phi_2^{(0)} = \phi_0 + \phi_1, \quad h_2^{(0)} = h_0 + h_1, \tag{4.6}$$

where the quantities u_1, ϕ_1, h_1 tend to zero exponentially outside the boundary layer.

We need first of all to calculate the right-hand side of (4.4) *in the outer region*. From (3.1), (3.2) and (3.12) we have that, in dimensional variables,

$$\mathbf{u}_1 \approx \mathrm{Re}\,[r\omega U_0(z)\,e^{i\omega t}]\,\hat{\boldsymbol{\theta}} = \mathrm{Re}\left[r\omega\frac{\cosh\Omega(1-z/d)}{\cosh\Omega}\,e^{i\omega t}\right]\hat{\boldsymbol{\theta}} \tag{4.7}$$

in the outer region. Hence it is easy to show that

$$[\mathbf{u}_1 . \nabla\mathbf{u}_1]_0^{(0)} = -\tfrac{1}{2}r\omega^2 U_0\,\bar{U}_0\,\hat{\mathbf{r}}, \tag{4.8}$$

where the overbar denotes complex conjugate. Next we have from (2.5) and (2.6) that

$$[\mathbf{A}_1(s)]_0 = r\omega\,.\,\mathrm{Re}\,[e^{i\omega(t-s)}\,U_0']\,(\hat{\boldsymbol{\theta}}\hat{\mathbf{z}}+\hat{\mathbf{z}}\hat{\boldsymbol{\theta}}) \tag{4.9}$$

and

$$[\boldsymbol{\xi}_1(s)]_0 = r\,.\,\mathrm{Re}\,[e^{i\omega t}(1-e^{-i\omega s})\,iU_0]\,\hat{\boldsymbol{\theta}}. \tag{4.10}$$

From these we obtain

$$[(\boldsymbol{\xi}_1 . \nabla)\,\mathbf{A}_1+\mathbf{A}_1 . \nabla\boldsymbol{\xi}_1+\nabla\boldsymbol{\xi}_1^T . \mathbf{A}_1]_0^{(0)} = -r^2\omega\sin\omega s\,U_0'\,\bar{U}_0'\,\hat{\mathbf{z}}\hat{\mathbf{z}} \tag{4.11}$$

and

$$[\mathbf{A}_1(s_1) . \mathbf{A}_1(s_2)]_0^{(0)} = \tfrac{1}{2}r^2\omega^2\cos\omega(s_1-s_2)\,U_0'\,\bar{U}_0'(\hat{\mathbf{z}}\hat{\mathbf{z}}+\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}). \tag{4.12}$$

Combining these expressions we now obtain

$$\mathbf{Z}_0^{(0)}(r,z) = r^2\omega^2 U_0'\,\bar{U}_0'[\beta_1\,\hat{\mathbf{z}}\hat{\mathbf{z}}+\tfrac{1}{2}\beta_2(\hat{\mathbf{z}}\hat{\mathbf{z}}+\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}})], \tag{4.13}$$

where we have defined

$$\beta_1 = -\int_0^\infty \zeta(s)\frac{\sin\omega s}{\omega}\,ds \tag{4.14}$$

and

$$\beta_2 = \int_0^\infty\int_0^\infty \gamma(s_1,s_2)\cos\omega(s_1-s_2)\,ds_1\,ds_2. \tag{4.15}$$

We assume now that the flow in the central region has the structure

$$\mathbf{u}_0(r,z) = u_0(r,z)\,\hat{\mathbf{r}}+w_0(r,z)\,\hat{\mathbf{z}}; \tag{4.16}$$

substituting this into equations (4.4), and using (4.8) and (4.13), we obtain

$$\mu_0\left[\frac{\partial^2 u_0}{\partial r^2}+\frac{1}{r}\frac{\partial u_0}{\partial r}-\frac{u_0}{r^2}+\frac{\partial^2 u_0}{\partial z^2}\right]-\frac{\partial\phi_0}{\partial r} = -\tfrac{1}{2}\rho r\omega^2 U_0\,\bar{U}_0+\tfrac{1}{2}\beta_2\,r\omega^2 U_0'\,\bar{U}_0', \tag{4.17}$$

$$\mu_0\left[\frac{\partial^2 w_0}{\partial r^2}+\frac{1}{r}\frac{\partial w_0}{\partial r}+\frac{\partial^2 w_0}{\partial z^2}\right]-\frac{\partial\phi_0}{\partial z} = -(\beta_1+\tfrac{1}{2}\beta_2)\,r^2\omega^2\frac{d}{dz}(U_0'\,\bar{U}_0'). \tag{4.18}$$

These have to be solved in conjunction with the continuity equation

$$\frac{\partial u_0}{\partial r}+\frac{u_0}{r}+\frac{\partial w_0}{\partial z} = 0. \tag{4.19}$$

A first approximation to the radial–axial flow field in the central region can be found in the similarity form

$$\begin{aligned} u(r,z) &= r\omega^2\psi'(z), \quad w(r,z) = -2\omega^2\psi(z),\\ \phi(r,z) &= \rho\omega^2[\sigma(z)+r^2\tau(z)]. \end{aligned} \tag{4.20}$$

These satisfy the continuity equation identically, while (4.17) and (4.18) reduce to

$$\rho\tau(z) = \rho k+(\beta_1+\tfrac{1}{2}\beta_2)\,U_0'\,\bar{U}_0' \tag{4.21}$$

and

$$\mu_0\,\psi''' = 2\rho k-\tfrac{1}{2}\rho U_0\,\bar{U}_0+(2\beta_1+\tfrac{3}{2}\beta_2)\,U_0'\,\bar{U}_0', \tag{4.22}$$

where k is a constant, together with a differential equation for $\sigma(z)$, namely

$$\rho\sigma' + 2\mu_0 \psi'' = 0. \tag{4.23}$$

We consider next the boundary conditions for the system (4.21)–(4.23). At the base of the cylinder we must have that $u_0 = w_0 = 0$, which become

$$\psi = 0, \quad \psi' = 0 \quad \text{on} \quad z = 0. \tag{4.24}$$

The tangential stress conditions at the free surface reduce to $w_0 = \partial u_0/\partial z = 0$, which become

$$\psi = 0, \quad \psi'' = 0 \quad \text{on} \quad z = d. \tag{4.25}$$

The normal stress conditions can be shown to reduce to

$$-\rho\omega^2[\sigma(d) + r^2\tau(d)] - 4\mu_0\omega^2\psi'(d) + \rho g h_0 = 0. \tag{4.26}$$

The side-wall conditions do not apply to the outer solution. We postpone until later consideration of the volume-conservation condition (2.17).

We first solve equation (4.22), using the boundary conditions (4.24) and (4.25) to determine the solution and the constant k. We set

$$M = \frac{(4\beta_1 + 3\beta_2)\,\omega}{|\mu|}, \quad p = \Omega + \bar{\Omega}, \quad iq = \Omega - \bar{\Omega}, \quad \eta = z/d. \tag{4.27}$$

Then with the aid of (3.12) we can write equation (4.22) as

$$(\mu_0/\rho d^3)\,\psi_{\eta\eta\eta} = 2k - \frac{(1-M)\cosh p(1-\eta) + (1+M)\cos q(1-\eta)}{2(\cosh p + \cos q)}. \tag{4.28}$$

Integrating and using the boundary conditions we find that

$$k = \frac{(1-M)\,[p^{-2}\cosh p - p^{-3}\sinh p] - (1+M)\,[q^{-2}\cos q - q^{-3}\sin q]}{\frac{4}{3}(\cosh p + \cos q)} \tag{4.29}$$

and

$$(\mu_0/\rho d^3)\,\psi_\eta = k(\eta^2 - 2\eta) + \frac{(1-M)p^{-2}[\cosh p - \cosh p(1-\eta)] - (1+M)q^{-2}[\cos q - \cos q(1-\eta)]}{2(\cosh p + \cos q)}. \tag{4.30}$$

Equation (4.26) can obviously be written in the form

$$\rho g h_0 = \rho\omega^2\left[\sigma(1) + r^2\tau(1) + \frac{4\mu_0}{\rho d}\psi_\eta(1)\right], \tag{4.31}$$

and from (4.21) we see that $\tau(1) = 1$. Hence (4.31) gives

$$h_0(r) = (\omega^2/g)\,[\sigma(1) + (4\mu_0/\rho d)\,\psi_\eta(1) + kr^2], \tag{4.32}$$

where the remaining constant $\sigma(1)$ has to be determined from the volume-conservation condition (2.17).

Before proceeding we consider the influence of the flow in the boundary layer. We see from (4.4) that the components $\mathbf{u}_1$, ϕ_1 of the velocity and pressure fields are driven by the self-interactions of those components of the first-order field which vanish outside the boundary layer. In other words, they satisfy a reduced version of (4.4), namely

$$\mu_0\nabla^2\mathbf{u}_1 - \nabla\phi_1 = [\rho(\mathbf{u}_1 . \nabla u_1)^{(0)} - \nabla . \mathbf{Z}^{(0)}]_1, \tag{4.33}$$

and the continuity equation. The velocity $\mathbf{u}_1$ is required to vanish at the base of the cylinder and at the side wall. The stress conditions at the free surface become equations relating $\mathbf{u}_1$, ϕ_1 and the height h_1 to known quantities which vanish outside the boundary layer. Finally, $\mathbf{u}_1$, ϕ_1 and h_1 are required to tend to zero exponentially outside the boundary layer.

Although the details are tedious and complicated, and will be omitted, it is conceptually fairly easy to see that the boundary-value problem just described does have a solution of the desired character. In particular we can show that

$$h_2^{(0)}(r) = h_0(r) + h_1(r), \tag{4.34}$$

where $h_0(r)$ is given by (4.32) and where $h_1(r)$ is significant only in the side-wall boundary layer.

We return to the volume-conservation condition (2.17). For the mean second-order flow this condition is

$$\int_0^a r h_2^{(0)}(r)\,dr = \int_0^a r h_0(r)\,dr + O(\lambda) = 0, \tag{4.35}$$

where the term $O(\lambda)$ comes from the integration of $rh_1(r)$, which is an exponentially decreasing function of $(a-r)/d$. Hence, applying (4.35) to the solution (4.32), we obtain to a first approximation the value of $\sigma(1)$, namely

$$\sigma(1) + (4\mu_0/\rho d)\,\psi_\eta(1) = -\tfrac{1}{2}ka^2. \tag{4.36}$$

Hence we deduce the expression

$$h_0(r) = (\omega^2 k/g)\,(r^2 - \tfrac{1}{2}a^2) \tag{4.37}$$

for the free surface height in the central region.

It is interesting to observe that, to this order of approximation, the free surface has the shape of a parabola (as it does in the case of rigid body rotation about a vertical axis). However, the parabola may be concave upward or downward, depending on the sign of the constant k. We shall discuss this question in detail in the next section.

The validity of our approximation scheme depends on the smallness of the parameter ϵ. Two separate factors contribute to an estimation of how small ϵ must be. First, the expansion (2.1) imposes a bound on the rate of strain, and some straightforward calculations lead to the estimate

$$\epsilon|\Omega \tanh \Omega|\, a/d \ll 1. \tag{4.38}$$

This condition validates the linearization of the equations of motion implicit in our procedure. Secondly, and independently, there is the assumption that the free surface boundary conditions could be imposed at the undisturbed position $z = d$. From (4.37) we see that this can be justified provided

$$\epsilon^2\omega^2 a^2|k|/gd \ll 1. \tag{4.39}$$

The estimates (4.38) and (4.39) taken together show that the admissible size of ϵ decreases as the frequency ω increases.

Finally we remark that equations (4.29) and (4.37) suggest a possible method for 'free surface viscometry' (cf. Joseph & Beavers, 1976). The only second-order material parameter involved in (4.29) is $(4\beta_1 + 3\beta_2)$, and the dependence is linear. Therefore if

μ is known for a fluid at a particular frequency, a measurement of the steady component of the free surface curvature for oscillations at a known small amplitude will at once enable $(4\beta_1 + 3\beta_2)$ to be evaluated.

5. Radial–axial flow and surface curvature

In this section we determine the direction of the steady streaming flow and the sign of the surface curvature. We are particularly interested in comparing non-Newtonian behaviour with Newtonian behaviour, and consequently we begin by describing the latter.

The direction of flow and the free surface curvature can be inferred from the results of the preceding section by taking $\mu = \mu_0$ and setting $M = 0$. Then we find that

$$p = q = (2\omega\rho d^2/\mu_0)^{\frac{1}{2}} \tag{5.1}$$

and (4.29) becomes

$$k = \frac{p(\cosh p - \cos p) - (\sinh p - \sin p)}{\frac{4}{3}p^3(\cosh p + \cos p)}. \tag{5.2}$$

It is easily verified that $k > 0$ for all p, that is, for all values of the frequency. Hence we conclude that the free surface in a Newtonian fluid is always concave upward, which is the anticipated result. From (5.2) we can show that $k \to \frac{1}{4}$ as $p \to 0$ and $k \sim 3/(4p^2)$ as $p \to \infty$. In view of (4.37) and (5.1) this means that

$$h_0(r) \approx \frac{\omega^2}{4g}(r^2 - \tfrac{1}{2}a^2) \quad \text{as} \quad \omega \to 0 \tag{5.3}$$

and

$$h_0(r) \sim \frac{3\mu_0\omega}{8\rho g d^2}(r^2 - \tfrac{1}{2}a^2) \quad \text{as} \quad \omega \to \infty. \tag{5.4}$$

(The latter result, of course, can only be interpreted in a limited sense because of the restrictions (4.38) and (4.39).)

Next, if we define a dimensionless radial velocity component by

$$\Psi_\eta = \frac{\mu_0}{\rho d^3}\psi_\eta, \tag{5.5}$$

we see that in the Newtonian case (4.30) becomes

$$\Psi_\eta = k(\eta^2 - 2\eta) + \frac{[\cosh p - \cosh p(1-\eta)] - [\cos p - \cos p(1-\eta)]}{2p^2(\cosh p + \cos p)}. \tag{5.6}$$

Straightforward analysis of (5.6) shows that $\Psi_\eta > 0$ for η near zero and $\Psi_\eta < 0$ for η near 1. This means that there is radial outflow of the steady streaming near the base of the cylinder and radial inflow near the free surface. Moreover, it can easily be shown that Ψ_η changes sign only once in the interval $0 < \eta < 1$. The position of the point where the change of sign occurs depends on p and moves towards the base as p increases. The variation of Ψ_η with η for various values of p has been computed, and some typical results are shown in figure 1.

We consider next the behaviour of a non-Newtonian fluid in the case of low-frequency oscillations. We define

$$\alpha_1 = -\int_0^\infty s\zeta(s)\,ds; \tag{5.7}$$

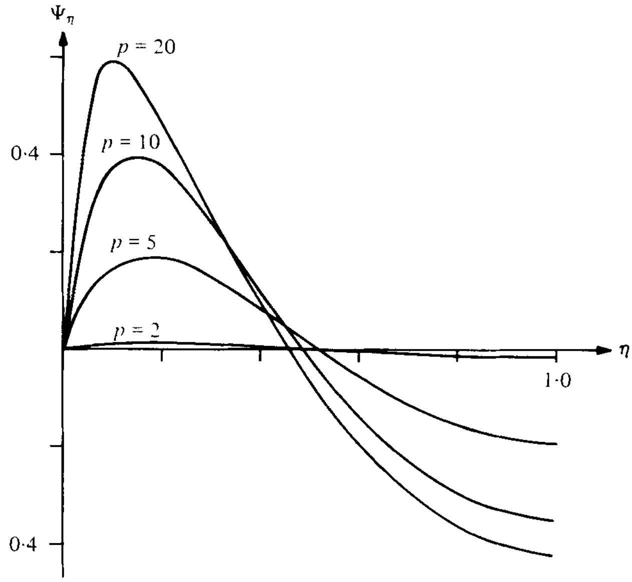

FIGURE 1. The radial component of the steady streaming flow for a Newtonian fluid at several values of frequency.

this material parameter is known to be negative ($\alpha_1 < 0$) in normal circumstances. Next we define the dimensionless parameters

$$\theta = -\frac{\omega\alpha_1}{\mu_0}, \quad c^2 = \frac{-4\rho d^2}{\alpha_1}. \tag{5.8}$$

θ is the Deborah number relative to the time-scale of the oscillations, and $c^2\theta$ is a Reynolds number. In the present context low frequency oscillations are defined by the inequalities

$$\theta \ll 1, \quad c^2 = O(1). \tag{5.9}$$

Expanding in powers of θ we obtain

$$\mu = \mu_0[1 - i\theta + O(\theta^2)], \quad q = (c^2\theta/2)^{\frac{1}{2}}[1 + \tfrac{1}{2}i\theta + O(\theta^2)]. \tag{5.10}$$

from which we deduce that

$$p = (\tfrac{1}{2}c^2\theta)^{\frac{1}{2}}[1 - \tfrac{1}{2}\theta + O(\theta^2)], \quad q = (\tfrac{1}{2}c^2\theta)^{\frac{1}{2}}[1 + \tfrac{1}{2}\theta + O(\theta^2)]. \tag{5.11}$$

Also, we can show that

$$\beta_1 = \alpha_1[1 + O(\theta^2)], \quad \beta_2 = \alpha_2[1 + O(\theta^2)], \tag{5.12}$$

where

$$\alpha_2 = \int_0^\infty \int_0^\infty \gamma(s_1, s_2)\, ds_1\, ds_2. \tag{5.13}$$

From (5.12) it follows that

$$M = -4\theta(1 - \tfrac{3}{4}\Lambda)[1 + O(\theta^2)], \tag{5.14}$$

where

$$\Lambda \equiv -\alpha_2/\alpha_1. \tag{5.15}$$

Substituting these approximations into (4.29) we obtain

$$k = \tfrac{1}{4}[1 + O(\theta^2)]. \tag{5.16}$$

Hence k is positive for oscillations of sufficiently low frequency, and this means that the parabola is concave upwards, as in the Newtonian case.

Next, we find that (4.30) becomes to leading order

$$\Psi_\eta = \frac{c^4\theta^2}{8.7!}(1-\bar\eta^2)[7\bar\eta^4+7\bar\eta^2-2+A(5\bar\eta^2-1)], \tag{5.17}$$

where
$$\bar\eta = 1-\eta, \quad A = \frac{252(1-\Lambda)}{c^2}. \tag{5.18}$$

The Newtonian case corresponds to $\alpha_1 = \alpha_2 = 0$, and hence to $A = 0$. For convenience we define: *Newtonian behaviour*, for which the flow is outwards near the bottom of the vessel and inwards at the free surface; *anti-Newtonian behaviour*, for which these directions are reversed; and *intermediate behaviour*, for which the flow has the same direction near both boundaries. These three categories are not sufficient to characterize the flow since, as we shall see below, there remains the question of the behaviour in the interior. However, they are convenient categories for the present.

We find from (5.17) that the flow is determined by the value of A, and an easy calculation gives the following results:

$A \geqslant -2$: the behaviour is Newtonian;
$A \leqslant -3$: the behaviour is anti-Newtonian;
$-3 < A < -2$: the flow has intermediate behaviour, being directed outwards near both boundaries, and balanced by a central inflow.

The quantity Λ is the ratio of two of the well-known Rivlin–Ericksen material constants, and in practice $\Lambda > 1$ for some real materials. Hence anti-Newtonian behaviour can occur (for suitable values of c) in the low frequency limit. This effect is due to the presence of the free surface; in the configuration studied by Rosenblat (1978) the steady streaming was shown to have Newtonian behaviour at low frequencies.

To exhibit the behaviour at *high frequencies* ($\omega \to \infty$) we require certain asymptotic forms. Integrating (3.5) by parts we find

$$\mu = -i\omega^{-1}\zeta(0) - \omega^{-2}\zeta'(0) + O(\omega^{-3}). \tag{5.19}$$

Then (3.7) and (4.27) give

$$p = -\rho^{\frac{1}{2}}d\zeta^{-\frac{3}{2}}(0)\,\zeta'(0) + O(\omega^{-2}), \quad q = 2\omega\rho^{\frac{1}{2}}d\zeta^{-\frac{1}{2}}(0) + O(\omega^{-1}). \tag{5.20}$$

Similarly we obtain

$$\beta_1 = -\omega^{-2}\zeta(0) + O(\omega^{-4}), \quad \beta_2 = \omega^{-2}\gamma(0,0) + O(\omega^{-2}), \tag{5.21}$$

whereupon (4.27) gives

$$M = \frac{3\gamma(0,0)}{\zeta(0)} - 4 + O(\omega^{-2}). \tag{5.22}$$

A simple application of these asymptotic forms to (4.39) and (4.30) gives

$$k = \frac{3(1-M)[p^{-2}\cosh p - p^{-3}\sinh p]}{4(\cosh p + \cos q)} + O(\omega^{-2}) \tag{5.23}$$

and
$$\Psi_\eta = (\omega^2/d)\left[\frac{(1-M)f(\eta)}{\cosh p + \cos q} + O(\omega^{-2})\right], \tag{5.24}$$

where

$$f(\eta) \equiv \tfrac{3}{4}(p^{-2}\cosh p - p^{-3}\sinh p)(\eta^2 - 2\eta) + \tfrac{1}{2}p^{-2}[\cosh p - \cosh p(1-\eta)]. \tag{5.25}$$

The quantity $p^{-2}\cosh p - p^{-3}\sinh p$ is positive for all p; hence the curvature coefficient k is positive if $M < 1$ and negative if $M > 1$. The function $f(\eta)$ can be shown by elementary methods to have the form of a Newtonian-type profile for all values of p. Thus the behaviour is Newtonian or anti-Newtonian according as $M < 1$ or $M > 1$ respectively. From (5.22) we deduce that, in the high frequency limit, the flow will be anti-Newtonian and the free surface curvature negative if and only if

$$\frac{\gamma(0,0)}{\zeta(0)} > \frac{5}{3}. \tag{5.26}$$

This criterion for anti-Newtonian behaviour was also obtained in Rosenblat's (1978) problem.

To examine the behaviour of the flow profile Ψ_η and the coefficient k over the whole range of frequencies it is necessary to propose explicit forms for the material functions $\zeta(s)$ and $\gamma(s_1, s_2)$. So as to gain some insight into this behaviour, we choose the simplest Maxwell-type approximations, namely

$$\zeta(s) = -(\mu_0^2/\alpha_1)\exp(\mu_0 s/\alpha_1) \quad (\alpha_1 < 0) \tag{5.27}$$

and

$$\gamma(s_1, s_2) = \alpha_2 \delta^2 \exp[-\delta(s_1+s_2)] \quad (\delta > 0). \tag{5.28}$$

These are consistent with (4.5), (5.7) and (5.13).

We find now that

$$p = c\theta^{\frac{1}{2}}(1+\theta^2)^{\frac{1}{4}}\cos\chi, \quad q = c\theta^{\frac{1}{2}}(1+\theta^2)^{\frac{1}{4}}\sin\chi \tag{5.29}$$

where

$$\chi = \tfrac{1}{2}\arctan\theta + \tfrac{1}{4}\pi; \tag{5.30}$$

moreover,

$$M = \theta(1+\theta^2)^{\frac{1}{2}}\left(\frac{3\Lambda\kappa^2}{\kappa^2+\theta^2} - \frac{4}{1+\theta^2}\right), \tag{5.31}$$

where

$$\kappa = -\alpha_1\delta/\mu_0. \tag{5.32}$$

With the aid of these relations we can calculate the quantities k and Ψ_η in terms of the four parameters θ, c, κ^2 and Λ. Of these parameters, only θ involves the frequency: it is the ratio of the elastic relaxation time of the material to the period of forced oscillations (the Deborah number). The parameters Λ and κ^2 involve only ratios of material quantities; both of these can be regarded as ratios of characteristic times. The parameter c generally occurs in the combination $c^2\theta$ which, as noted earlier, has the form of a Reynolds number.

We consider first the conditions under which the free surface curvature coefficient k changes sign. For fixed values of Λ and c, setting $k = 0$ divides the $\kappa^2-\theta$ plane into regions $k > 0$ and $k < 0$. Figure 2 depicts the situation as computed numerically for $\Lambda = 1{\cdot}5$ (a physically realistic value for certain materials) and for the values $c = 1$, 2 and 10. We see that $k > 0$ whenever θ is small, which agrees with our small frequency approximation. Also, $k < 0$ for large θ and large κ^2 is small. The computations show that the transition value of κ^2 at which k changes sign for large θ is approximately $\kappa^2 = 1{\cdot}1$, independent of c. Now from (5.27) and (5.28) we have that

$$\frac{\gamma(0,0)}{\zeta(0)} = \Lambda\kappa^2, \tag{5.33}$$

and when $\Lambda = 1{\cdot}5$ the condition (5.26) is equivalent to $\kappa^2 > \frac{10}{9}$. Hence there is excellent agreement between the numerical and the asymptotic results.

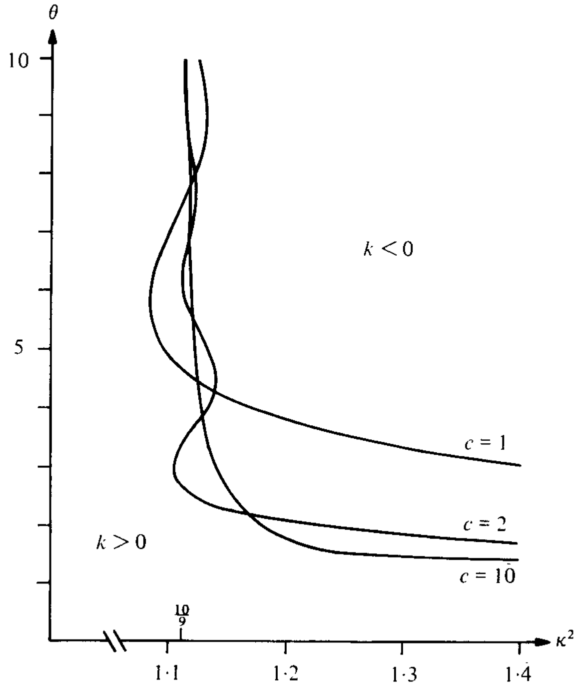

FIGURE 2. Points of reversal of the free-surface curvature in a non-Newtonian fluid for $\Lambda = 1{\cdot}5$ and several values of c. $k > 0$: surface is concave upwards; $k < 0$: surface is concave downwards.

The dependence of the velocity profile Ψ'_η on the parameters θ, c, κ^2 and Λ is much more complicated. Generally speaking, for fixed values of c and Λ there is a single curve in the $\kappa^2 - \theta$ plane which separates a regime where the flow near the free surface is outward from a regime where this flow is inwards. This curve is close to, but does not coincide with, the curve separating regions of positive and negative free surface curvature in the $\kappa^2 - \theta$ plane for the same values of c and Λ.

The behaviour near the lower boundary is very complex. Instead of just one separatrix there are many curves dividing the plane into regions of positive and negative outflow. Thus, as θ varies with κ^2 fixed, the direction of the radial flow at the lower boundary can switch back and forth several times; the condition (5.26) is clearly only a high frequency result. A typical situation is shown in figure 3, which corresponds to the case $\Lambda = 1{\cdot}5$ and $c = 1$. There are regions of Newtonian, anti-Newtonian and intermediate flow, in the sense defined above. There, are, moreover, two types of intermediate flow: one in which the motion is radially outwards near both boundaries, and one in which it is radially inwards. Even this, however, does not tell the whole story, since there can be several changes of direction of the flow in the interior for each fixed set of parameter values. The profiles can be computed quite easily, but we have not thought it worthwhile to do so here.

To summarize, we find that reversal of the steady flow is possible at any frequency, but that reversal of the free surface curvature cannot occur at low frequencies. Under certain conditions the free surface curvature, the flow at the free surface and, most strikingly, the flow at the lower boundary may undergo several reversals as the frequency is increased.

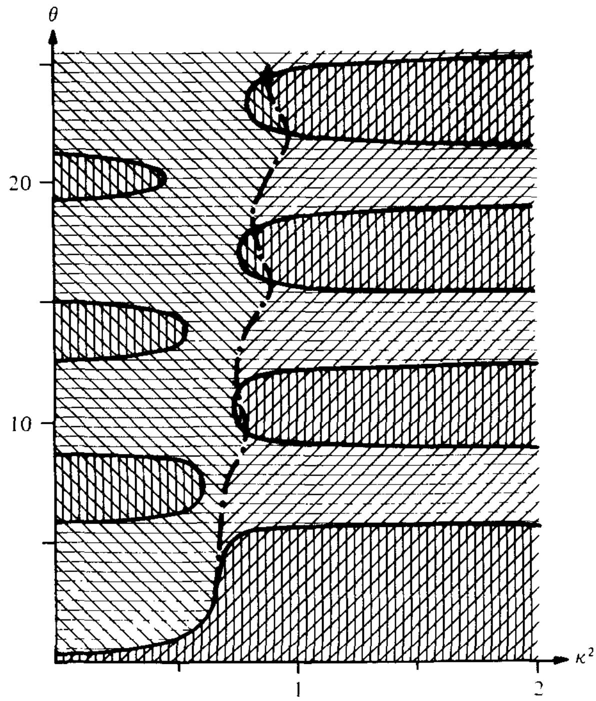

FIGURE 3. Regions of Newtonian, anti-Newtonian and intermediate behaviour of the steady streaming for $\Lambda = 1{\cdot}5$ and $c = 1$. ▨, Outward radial flow near the free surface. ▧, Inward radial flow near the free surface. ▤, Outward radial flow near the base. ▥, Inward radial flow near the base.

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